EXAMPLE 3. In spherical coordinates a vector field is given by $\mathbf{A} = (5/r^2)\sin\theta\,\mathbf{a}_r + r\cot\theta\,\mathbf{a}_\theta + r\sin\theta\cos\phi\,\mathbf{a}_\phi$. Find div **A**.

$$\operatorname{div}\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(5\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin\theta\cot\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r\sin\theta\cos\phi) = -1 - \sin\phi$$

4.3 DIVERGENCE OF D

From Gauss' law (Section 3.3),

$$\frac{\oint \mathbf{D}\cdot d\mathbf{S}}{\Delta v} = \frac{Q_{enc}}{\Delta v}$$

In the limit,

$$\lim_{\Delta v\to 0}\frac{\oint \mathbf{D}\cdot d\mathbf{S}}{\Delta v} = \operatorname{div}\mathbf{D} = \lim_{\Delta v\to 0}\frac{Q_{enc}}{\Delta v} = \rho$$

This important result is one of Maxwell's equations for static fields:

$$\operatorname{div}\mathbf{D} = \rho \qquad \text{and} \qquad \operatorname{div}\mathbf{E} = \frac{\rho}{\epsilon}$$

if ϵ is constant throughout the region under examination (if not, $\operatorname{div}\epsilon\mathbf{E} = \rho$). Thus both **E** and **D** fields will have divergence of zero in any isotropic charge-free region.

EXAMPLE 4. In spherical coordinates the region $r \le a$ contains a uniform charge density ρ, while for $r > a$ the charge density is zero. From Problem 2.54, $\mathbf{E} = E_r\mathbf{a}_r$, where $E_r = (\rho r/3\epsilon_0)$ for $r \le a$ and $E_r = (\rho a^3/3\epsilon_0 r^2)$ for $r > a$. Then, for $r \le a$,

$$\operatorname{div}\mathbf{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\rho r}{3\epsilon_0}\right) = \frac{1}{r^2}\left(3r^2\frac{\rho}{3\epsilon_0}\right) = \frac{\rho}{\epsilon_0}$$

and, for $r > a$,

$$\operatorname{div}\mathbf{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\rho a^3}{3\epsilon_0 r^2}\right) = 0$$

4.4 THE DEL OPERATOR

Vector analysis has its own shorthand, which the reader must note with care. At this point a vector operator, symbolized ∇, is defined *in cartesian coordinates* by

$$\nabla \equiv \frac{\partial(\ )}{\partial x}\mathbf{a}_x + \frac{\partial(\ )}{\partial y}\mathbf{a}_y + \frac{\partial(\ )}{\partial z}\mathbf{a}_z$$

In the calculus a differential operator D is sometimes used to represent d/dx. The symbols $\sqrt{\ }$ and $\int$ are also operators; standing alone, without any indication of what they are to operate on, they look strange. And so ∇, standing alone, simply suggests the taking of certain partial derivatives, each followed by a unit vector. However, when ∇ is dotted with a vector **A**, the result is the divergence of **A**.

$$\nabla\cdot\mathbf{A} = \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right)\cdot(A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \operatorname{div}\mathbf{A}$$

Hereafter, the divergence of a vector field will be written $\nabla\cdot\mathbf{A}$.

Warning! The del operator is defined only in cartesian coordinates. When $\nabla \cdot \mathbf{A}$ is written for the divergence of $\mathbf{A}$ in other coordinate systems, it does not mean that a del operator can be defined for these systems. For example, the divergence in cylindrical coordinates will be written as

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

(see Section 4.2). This *does not imply that*

$$\nabla = \frac{1}{r}\frac{\partial}{\partial r}(r\quad)\mathbf{a}_r + \frac{1}{r}\frac{\partial(\quad)}{\partial \delta}\mathbf{a}_\phi + \frac{\partial(\quad)}{\partial z}\mathbf{a}_z$$

in cylindrical coordinates. In fact, the expression would give *false results* when used in ∇V (the gradient, Chapter 5) or $\nabla \times \mathbf{A}$ (the curl, Chapter 9).

4.5 THE DIVERGENCE THEOREM

Gauss' law states that the closed surface integral of $\mathbf{D} \cdot d\mathbf{S}$ is equal to the charge enclosed. If the charge density function ρ is known throughout the volume, then the charge enclosed may be obtained from an integration of ρ throughout the volume. Thus,

$$\oint \mathbf{D} \cdot d\mathbf{S} = \int \rho\, dv = Q_{\text{enc}}$$

But $\rho = \nabla \cdot \mathbf{D}$, and so

$$\oint \mathbf{D} \cdot d\mathbf{S} = \int (\nabla \cdot \mathbf{D})\, dv$$

This is the *divergence theorem*, also known as *Gauss' divergence theorem*. It is a three-dimensional analog of Green's theorem for the plane. While it was arrived at from known relationships among $\mathbf{D}$, Q, and ρ, the theorem is applicable to any sufficiently regular vector field.

divergence theorem $$\oint_S \mathbf{A} \cdot d\mathbf{S} = \int_v (\nabla \cdot \mathbf{A})\, dv$$

Of course, the volume v is that which is enclosed by the surface S.

EXAMPLE 5. The region $r \le a$ in spherical coordinates has an electric field intensity

$$\mathbf{E} = \frac{\rho r}{3\epsilon}\mathbf{a}_r$$

Examine both sides of the divergence theorem for this vector field. For S, choose the spherical surface $r = b \le a$.

$$\oint \mathbf{E} \cdot d\mathbf{S}$$

$$\iint \left(\frac{\rho b}{3\epsilon}\mathbf{a}_r\right) \cdot (b^2 \sin\theta\, d\theta\, d\phi\, \mathbf{a}_r)$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{\rho b^3}{3\epsilon}\sin\theta\, d\theta\, d\phi$$

$$= \frac{4\pi\rho b^3}{3\epsilon}$$

$$\int (\nabla \cdot \mathbf{E})\, dv$$

$$\nabla \cdot \mathbf{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\rho r}{3\epsilon}\right) = \frac{\rho}{\epsilon}$$

then $$\int_0^{2\pi}\int_0^{\pi}\int_0^{b} \frac{\rho}{\epsilon} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{4\pi\rho b^3}{3\epsilon}$$

The divergence theorem applies to time-varying as well as static fields in any coordinate